Transformation of the Christoffel Symbols

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In this note, I want to elaborate on covariant derivatives and Christoffel symbols. In particular, I want to explain why Christoffel symbols are not tensors, and how we can understand their transformation properties under general coordinate transformations.

As we have discussed numerous times before, the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ are objects that we introduced to account for the fact that the ordinary derivative ∂_μ does not transform like a tensor on curved spacetime manifolds. Instead, we have to define a new covariant derivative ∇_μ, which acts on a general rank (m, n) tensor T as follows:

$$\begin{aligned}\nabla_\mu T^{\nu_1\ldots\nu_m}{}_{\rho_1\ldots\rho_n} &= \partial_\mu T^{\nu_1\ldots\nu_m}{}_{\rho_1\ldots\rho_n} \\ &+ \Gamma^{\nu_1}_{\mu\lambda} T^{\lambda\nu_2\ldots\nu_m}{}_{\rho_1\ldots\rho_n} + \Gamma^{\nu_2}_{\mu\lambda} T^{\nu_1\lambda\ldots\nu_m}{}_{\rho_1\ldots\rho_n} + \ldots + \Gamma^{\nu_n}_{\mu\lambda} T^{\nu_1\ldots\nu_{m-1}\lambda}{}_{\rho_1\ldots\rho_n} \\ &- \Gamma^{\lambda}_{\mu\rho_1} T^{\nu_1\ldots\nu_m}{}_{\lambda\rho_2\ldots\rho_n} - \Gamma^{\lambda}_{\mu\rho_2} T^{\nu_1\ldots\nu_m}{}_{\rho_1\lambda\ldots\rho_n} - \ldots - \Gamma^{\lambda}_{\mu\rho_m} T^{\nu_1\ldots\nu_m}{}_{\rho_1\ldots\rho_{n-1}\lambda} \ .\end{aligned} \tag{1}$$

That is, there is a piece of that looks like the ordinary partial derivative acting on T, as well as one term with a Christoffel symbol for each index on T. Note that the sign in front of the Christoffel symbol depends on whether you have upper or lower indices. Each individual Christoffel symbol is not a tensor, nor is the partial derivative, but combined in the way written above, the entire sum is a tensor.

When we say that something is a tensor, we mean that it transforms a particular way that is compatible with the Lorentz group under coordinate transformations. In particular, for a rank (m, n) tensor T, when we apply a coordinate transformation of the form $x^\mu \to x^{\mu'}$, the tensor T transforms correspondingly as

$$T^{\nu_1\ldots\nu_m}{}_{\rho_1\ldots\rho_n} \to T^{\nu'_1\ldots\nu'_m}{}_{\rho'_1\ldots\rho'_n} \ , \tag{2}$$

where the right-hand side is given by

$$T^{\nu'_1\ldots\nu'_m}{}_{\rho'_1\ldots\rho'_n} = \frac{\partial x^{\nu'_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\nu'_m}}{\partial x^{\nu_m}} \frac{\partial x^{\rho_1}}{\partial x^{\rho'_1}} \cdots \frac{\partial x^{\rho_n}}{\partial x^{\rho'_n}} T^{\nu_1\ldots\nu_m}{}_{\rho_1\ldots\rho_n} \ . \tag{3}$$

The Christoffel symbols are not tensors, and so this transformation law will not apply to them. However, this does beg the question: how do Christoffel symbols transform? They will still have some associated transformation under coordinate transformations, and it would be interesting to write it down and verify explicitly that it does not match the form of (3).

To establish the transformation properties of Christoffel symbols, let's consider a vector V, whose covariant derivative is given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\rho} V^\rho \ . \tag{4}$$

Since this covariant derivative is a tensor, we can use (3) to establish that it transforms as $\nabla_\mu \to \nabla_{\mu'} V^{\nu'}$, with

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \ . \tag{5}$$

The left-hand side of (5) can be expanded by writing the covariant derivative explicitly in terms of ordinary derivatives and Christoffel symbols, which results in

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\rho'} V^{\rho'} \ . \tag{6}$$

Then, since V is a tensor, we know how it transforms using (3), and so we can further express this as

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} \left(\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \right) + \Gamma^{\nu'}_{\mu'\rho'} \left(\frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} \right) \ . \tag{7}$$

Then, we can trade $\partial_{\mu'}$ for $\frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$ via the chain rule, and so we have

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \left(\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \right) + \Gamma^{\nu'}_{\mu'\rho'} \left(\frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} \right) \ . \tag{8}$$

Finally, we can use the product rule to to write out how the derivative ∂_{μ} can act either on $\frac{\partial x^{\nu'}}{\partial x^{\nu}}$ or V^{ν}, and so we are left with the nice result

$$\begin{aligned} \nabla_{\mu'} V^{\nu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\partial_{\mu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \right) V^{\nu} + \Gamma^{\nu'}_{\mu'\rho'} \left(\frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} \right) \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} V^{\nu} + \Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} \ , \end{aligned} \tag{9}$$

where the second equality came from just writing out $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ explicitly. Notice that (9) exactly matches equation (3.3) in Sean Carroll's lecture notes on relativity, which you can find at [https://arxiv.org/abs/gr-qc/9712019].

Ok, so we now know from (9) how the left-hand side of (5) can be written in terms of ordinary derivatives and Christoffel symbols, and with all instances of V appearing with ordinary (and not primed) indices. We now want to do a similar thing to the right-hand side of (5) by expanding out the covariant derivative explicitly:

$$\begin{aligned} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \nabla_{\mu} V^{\nu} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\rho} V^{\rho} \right) \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\rho} V^{\rho} \ . \end{aligned} \tag{10}$$

Equations (9) and (10) should both be equal, since they are the left- and right-hand sides of (5), respectively. Setting them equal results in the following equation:

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} V^{\nu} + \Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\rho} V^{\rho} \ . \tag{11}$$

The first term on the left-hand side and the first term on the right-hand side of (11) both match, and so they can be cancelled out. If we additionally relabel dummy indices on V such that all instances of V appear as V^{ρ}, then (11) becomes

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\rho}} V^{\rho} + \Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\rho} V^{\rho} \ , \tag{12}$$

or equivalently by reorganizing terms we can write this as

$$\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\rho}} + \Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^{\rho}} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\rho} \right) V^{\rho} = 0 \ . \tag{13}$$

Crucially, we have thus far assumed nothing about the vector field V. This means that (13) must hold for any arbitrary vector field, including ones that are non-zero everywhere, and so the only way for (13) to be true in general is for the term in parentheses to vanish identically. Therefore we must have

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\rho}}+\Gamma^{\nu'}_{\mu'\rho'}\frac{\partial x^{\rho'}}{\partial x^{\rho}}-\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\rho}=0\ . \tag{14}$$

We're now almost done. We just need to isolate the term with $\Gamma^{\nu'}_{\mu'\rho'}$ entirely. We will do this by multiplying everything by $\frac{\partial x^{\rho}}{\partial x^{\sigma'}}$, since we know that

$$\frac{\partial x^{\rho'}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x^{\sigma'}}=\delta^{\rho'}_{\sigma'}\ , \tag{15}$$

i.e. these two derivatives are inverses of one another if the indices ρ' and σ' match. Thus, we end up with

$$\frac{\partial x^{\rho}}{\partial x^{\sigma'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\rho}}+\Gamma^{\nu'}_{\mu'\sigma'}-\frac{\partial x^{\rho}}{\partial x^{\sigma'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\rho}=0\ . \tag{16}$$

Relabelling the free index σ' to ρ' rearranging this slightly, we end up with the following transformation property of the Christoffel symbols:

$$\boxed{\Gamma^{\nu'}_{\mu'\rho'}=\frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\rho}-\frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\rho}}}\ . \tag{17}$$

When we compare this final result (17) to the tensor transformation law (3), we can see that the two do not match. The first term on the right-hand side of (17) is the expected tensor transformation law, but then we have this additional piece that is not proportional to any Christoffel symbols and just requires taking lots of derivatives of coordinates. Thus, we have explicitly proven that the Christoffel symbols do not transform as tensors, though we can nonetheless obtain an exact result (17) for how they do transform.